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Images of numbers, or "When 98 is upper left and 6 sky blue"*

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Abstract

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Some people declare that they possess a personal visual representation of numbers: some automatically "see" the numbers they are confronted with in a precise location in a structured mental space, others "associate" specific colours with given numbers. Such visuo-spatial representations of numbers were first described by Galton in 1880 but have since received little attention from psychologists. It is the aim of this article to describe these mental representations and discuss their role in number processing. The authors first review Galton's observations, and then present their own. Finally, they discuss the relevance of these visuo-spatial representations in relation to contemporary debates on number representation and calculation.

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GALTON'S CONTRIBUTION

In 1880 Galton published two articles (Galton, 1880a, 1880b) describing visuo-spatial number representations which he had collected from 80 male and female subjects. Galton distinguished those representations from the capacity to mentally visualize in an effortful way a number or an arithmetic operation as it would appear if actually written. The kind of number representation Galton's subjects referred to is more complex and is automatically activated when they see, hear or think of a number.

Some number representations concern simple digits or numbers. For example, subjects reported seeing numbers as they appear on dominoes or on playing cards; others declared they saw digits in a specific colour. Other subjects reported seeing each number at a given place in a stable spatial mental structure. When these subjects thought of the series of numbers "they show themselves in a definite pattern that always occupies an identical position in their field of view with respect to the direction in which they were looking" (Galton, 1883). These visual patterns, called by Galton the "number-form" (hereafter NF), consist either in a simple line with or without shifts in orientation (see Figure 1) or take more complex forms such as rows or grids. They may be coloured, present changes in luminosity at some locations or occupy different planes (see Figure 2).

Bertillon (1880, 1881) reported similar data, but added (Bertillon, 1882) that some subjects also had visuo-spatial representations for the months of the year or the days of the week (see Figure 3).

These NFs vary with respect to their structure (lines, grids, rows, changes in orientation and colour, location of the first number, etc.), but some common characteristics can be drawn.

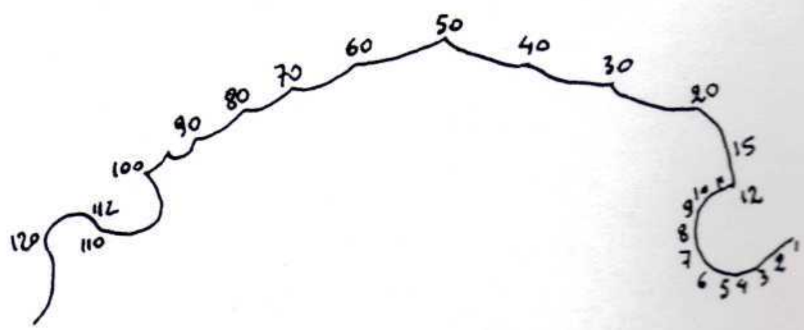

Figure 1. *Number-form with a clock-like form at the beginning (Galton, 1880a, reprinted by permission from* Nature *vol. 21, p. 253, Copyright © 1880 Macmillan Magazines Ltd.).*

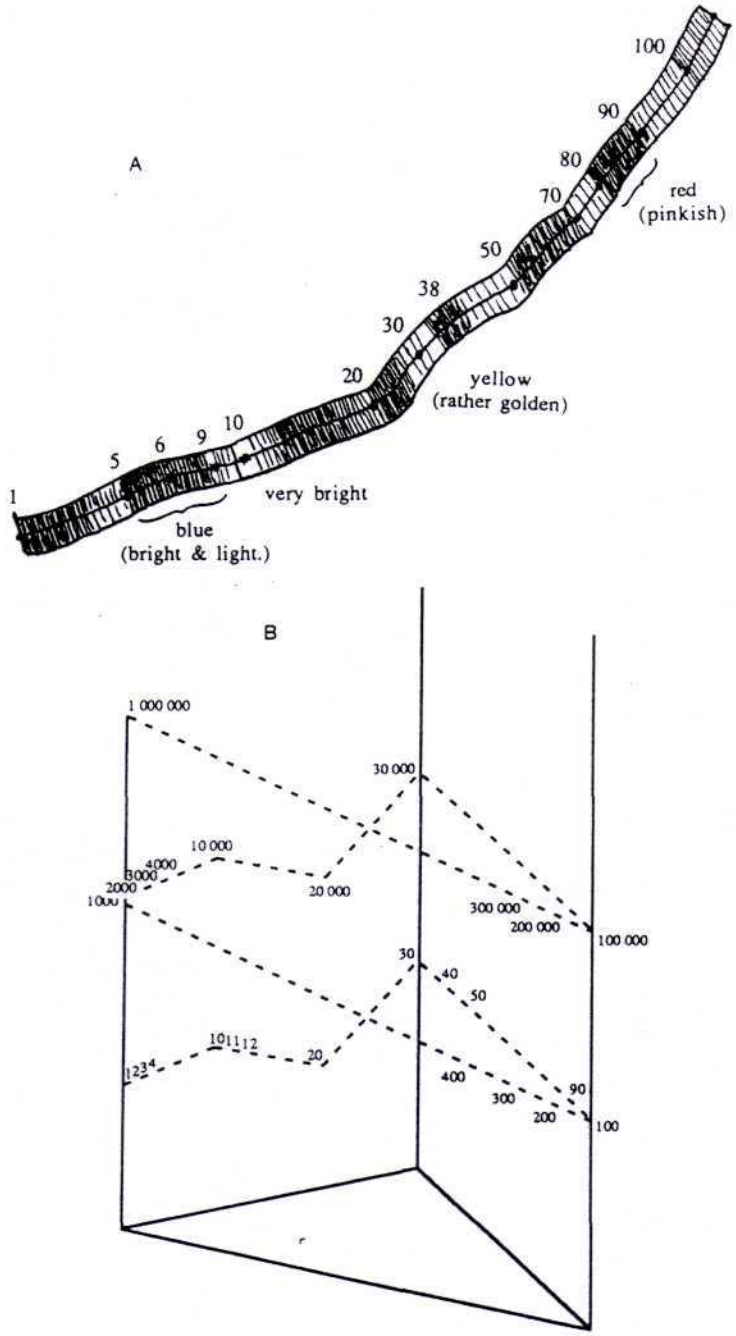

Figure 2. *Number-form: (A) with change in luminosity and colours at some location (Galton, 1880a, reprinted by permission from* Nature *vol. 21, p. 254, Copyright © 1880 Macmillan Magazines Ltd.) and (B) occupying different planes (Bertillon, 1882, p. 267).*

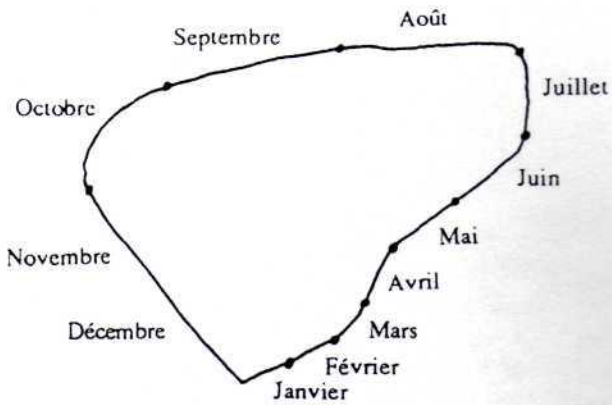

Figure 3. *Visuo-spatial representation for months on a deformed dial (Bertillon, 1882, p. 267).*

Firstly, there seemed to be a clear *intra-subject consistency*. For a given subject, the whole NF always had an identical structure, each number always occupying the same position on the NF, with invariant size and form characteristics. Similarly, the subjects having coloured number images always saw the same number associated with the same colour. A second common characteristic was the *emergence of the NF during infancy* with no evident relationship to any number and calculation teaching methods at school. A third characteristic of these NFs was the *mandatory and automatic aspect of their activation*. The subjects indicated that the NFs were automatically activated by any number that was heard, seen or came to mind. A last common characteristic was the *subdivision of the NF into a clear and precise part and a less precise part.*

Even though he did not carry out a systematic statistical analysis on the distribution of this particular ability, Galton estimated that it could be present in about 1 out of 30 adult males and 1 out of 15 females. The phenomenon is thus rare but surely not exceptional. Finally, there appeared to be no regular correlation between the possession of a number representation and the level of expertise in arithmetic even if some individuals claimed to use their NF at least for simple mental arithmetic operations.

The object of this article is twofold: firstly to confirm and document the observations of Galton by collecting new data about how some people mentally represent numbers; secondly to extend these introspective reports by examining with chronometric and experimental investigations two subjects presenting specific number representations in order to try to certify their existence and clarify some aspects of their role in number processing. Therefore, the present study only constitutes a preliminary and exploratory investigation in this strange and very specific aspect of human cognition.

A NEW INQUIRY

Procedure and material

Data collection

Informal inquiry

During the last three years, we asked people around us how they mentally visualized numbers. When they claimed to possess some particular number representation, we asked them to write a short description of their representations (and to draw it as precisely as possible), their functioning and conditions of use. Each subject was then given a more systematic and extended questionnaire about his or her number representation.

Systematic inquiry

At the same time, in order to collect more data, a short questionnaire was given to 194 psychology students at the University of Louvain (153 females and 41 males; mean age = 22 years 4 months; S.D. = 5.1). Those who claimed to possess particular number representations also received, in a second step, the more systematic extended questionnaire.

Description of the extended questionnaire

This questionnaire contained 20 main questions, most of them divided into sub-questions. Some were open-ended questions, others were of the "yes/no/don't know" type, and some required frequency-of-use judgments.

Different aspects of the number representation were examined: (1) its structure – shape, colour, clarity of the different parts, stability of these characteristics with time; (2) some functioning aspects – the "mode of entry" into the number representation, the subject's position in relation to it, the automaticity of its activation, its sensitivity to concurrent tasks (e.g., reading), effect (if any) of environmental modifications (e.g., dark) or subject's state (e.g., fatigue); (3) its use in diverse numerical activities (e.g., writing, hearing or thinking of numbers in various contexts such as money exchange, phone numbers, addresses, ages, historical dates, time of the day, etc.) or in calculation. As regards its use in calculation, a distinction was made between the different operations as well as simple/complex and mental/written operations. It was also attempted to evaluate whether the number representations were used to realize the operation or only to visualize some numbers (e.g., the data of the problem, some intermediate results); (4) its origin: conditions of emergence, influence of schooling, and the possible existence of a progressive evolution of the number representation; (5) the presence of other representations (for days, months, etc.); (6) and the existence of such representations in other members of the subject's family.

As almost all representations were of a visual type, subjects' degree of imagery was examined through Paivio's Individual Difference Questionnaire (IDQ) (Paivio, 1971; Paivio & Harschman, 1983). This questionnaire evaluates the extent to which people use verbal or visual strategies in information processing, problem solving and thinking in general. The IDQ provides two scores: one for habit in using verbal processes (verbalizer style) the other in using visual ones (imager style).

Results

Twenty-two spontaneous written introspective reports were collected through the informal inquiry, and the systematic inquiry yielded 27 more positive answers (21 females (13.7%) and 6 males (14.6%) out of 194 subjects). Among these 49 subjects, 26 accepted to answer the extended questionnaire. Our database is thus composed of 26 complete records and 23 partial records. The results presented here are derived from the analysis of the complete records. Twenty-two subjects were aged from 18 to 30 years (mean = 23; S.D. = 3.5); the remaining four were older than 35 years.

The structure of the number representations

Two main types of number representations were distinguished: continuous lines, scales, grids (all called NF), and coloured codes. However, associated images and simple analogical representations were also observed. All these number representations will be briefly described. Table 1 shows the distribution of these different types of number representation.[1]

Number forms

Fourteen *line* or *scale* NFs were observed. Here, the numbers are presented in their ordinal succession on graduated lines (Figure 4A), in strips divided into boxes (Figure 4B), or one after the other without any support (Figure 4C). These structures may be in a vertical or horizontal arrangement, sometimes rectilinear or containing shifts in orientation at some critical points (decade frontiers, or 100,

[1]Some subjects belonging to two groups, some giving several or no answers to some questions, so total frequencies of responses sometimes do not match number of subjects.

Table 1. *Distribution of the number representations from complete and partial records*

Type of number representation	Complete records	Partial records
Number-form: lines or scales	14	8
grids	3	1
Coloured codes, associated images, and synaesthesia	7[a]	6[a]
Analogical representations	3	4
Undefined type	–	5
Total	27[a]	24[a]

[a]Two subjects having both lines NF and coloured codes; the total of number representations exceeds the number of subjects.

1000, etc.) or at some step near the beginning of the scale (shifts at 5, 10 or 12, etc.). The numbers are always written in digits. In ten cases, they are handwritten (1 case typewritten, 3 unanswered questions). All their characteristics (shape, colour, size, and relative position) remain constant. The decade frontiers are more salient (the number could be blacker or more luminous, or the graduation more marked). In six cases, there also exist changes in colour either of the numbers themselves or of the background. The degree of distinctiveness varies: a first part, until 20 or 50, is complete and clear, while the part reaching 100 or beyond becomes less and less precise. Like Galton, we also found two subjects one of whom presented a clockwise and the other an anti-clockwise clock-like organization of the first 12 numbers.

Another type of NF are the *grids* (3 cases). Here, the numbers are also organized in a spatial structure, but after a linear development until 29 the numbers take their place in an ascending columns organization until 99. At this point the spatial organization is repeated, and other grids exist for numbers to 199, 299, etc. (see Figure 5). These subjects did not describe any variation of distinctiveness for the numbers belonging to the first grid. Except for this, the grids shared all the characteristics of the line or scale NF.

Coloured codes, associated images or synaesthesias

Seven subjects associated a colour with digits (1 subject also had a line NF). The digit itself (or in one case the number-word) is coloured (see Figure 6), but the association may also be related to the background, or to a synaesthesia that the subjects find more difficult to describe (one subject reported "having the feeling" of a colour). The associations usually exist for numbers below 10 or 12 only, and occasionally for the decades. The associations are generally very strong and the colour precise and nuanced, but in some cases only some digits present a

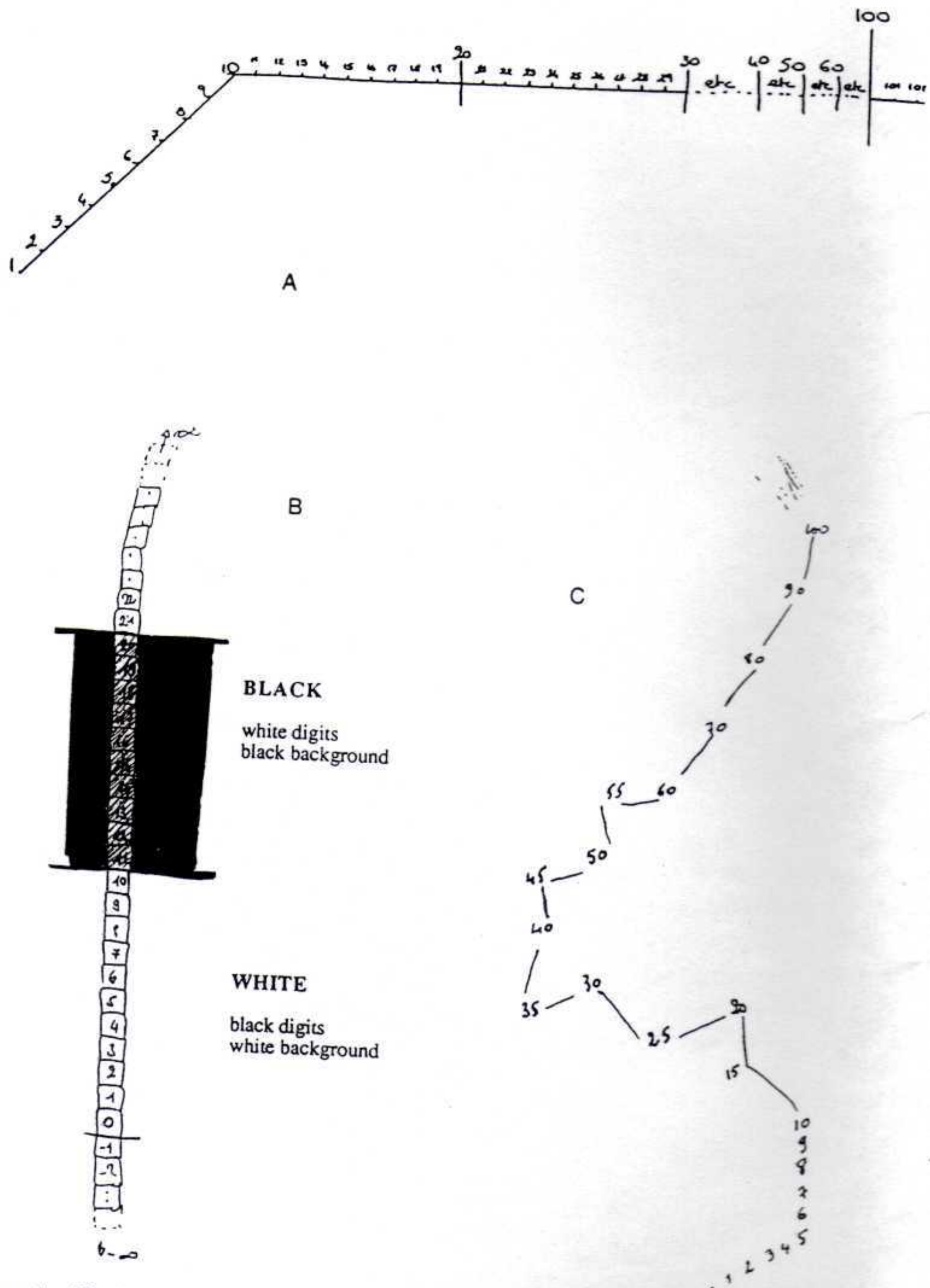

Figure 4. *Number-forms where numbers are presented in their ordinal order succession (A) on a graduate line, (B) in strip divided in boxes, and (C) without any support.*

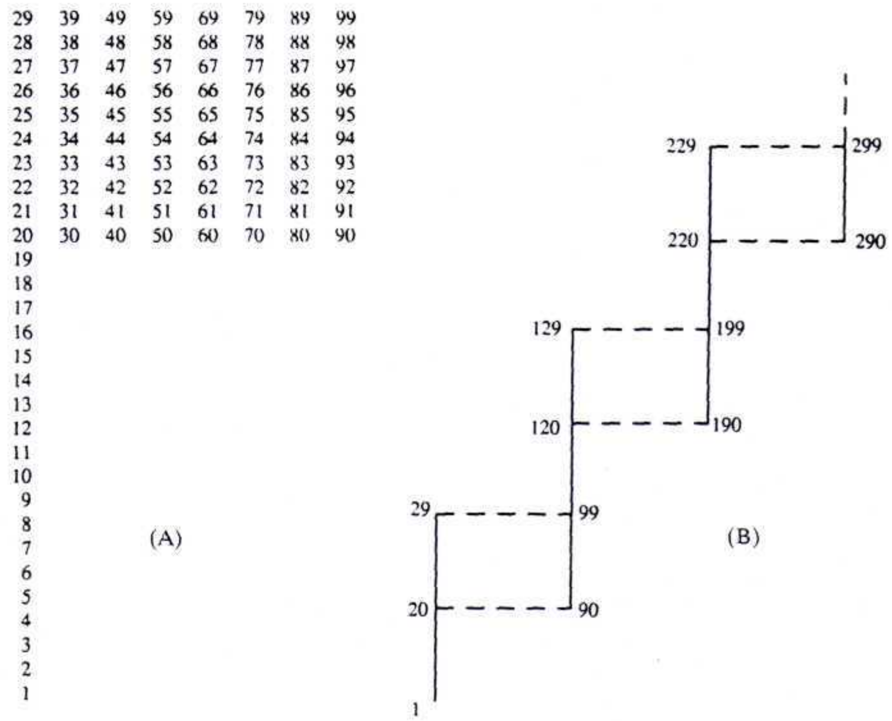

Figure 5. *Grid number-form for one subject: (A) from 1 to 99, and (B) global representation.*

clear and distinct colour, while the others are described as less "clear" or "dark-coloured", or as sharing a colour with another digit.

For three subjects, associations other than colours were also present. For example, geometrical figures having or not having analogical relations with the digit quantity (a square for 4) may represent the digits. Two subjects experienced various feelings in relation to numbers: numbers felt "hot" or "cold". For one subject, there was no systematicity behind this classification, while the other associated numbers' temperature with parity: odd numbers were "cold", while even were "warm and sympathic". Some associated images may be more precise

0 : white - colourless	5 : white
1 : black	6 : sky blue
2 : green	7 : pink-mauve
3 : red	8 : yellow
4 : orange	9 : grey

Figure 6. *Colours associates for one subject.*

and structured, for example for one subject, "hundred is a perfect circle, 8 is a big and cool person, 5 is someone running . . ." These astonishing image and feeling associations are close to those of the famous mnemonist described by Luria (1970).

Simple analogical representations

Three subjects described number representations which we called "analogical" because the quantity was directly represented by patterns of dots or other things such as alignment of apples, parts of a bar of chocolate, etc.

Functioning aspects

Regarding the *access to a specific number*, the subjects indicated that they immediately saw the number in its position in the NF. Some reports are, however, ambiguous: a number could be viewed in relation to various adjacent numbers. One subject, for example, described a sort of focusing mechanism in which the number is first viewed with neighbouring numbers and then it is focused from the adjacent ones. This activation is expressed by saying that "the number (or, a larger area) becomes more precise or salient". In the case of indirect access, subjects take advantage of more salient locations (multiples of 10 or 5) in their NF. For the coloured and analogical representations, the access to a specific number is always immediate.

Concerning *the position of the subject* in relation to his number representation, 19 subjects (12 from the NF group, and 7 from the coloured code and analogical groups) described being exterior to their representations and seeing them as if they were drawn on a sheet of paper in front of them. Only subjects of the NF group (5) indicated to be inside their representation, whereas one subject from the NF group and 3 from the coloured code group reported being able to move the representations before them.

Regarding the *mandatory and automatic aspect of the activation*, for 12 of the 26 cases (8 from the NF group) the representations automatically arise when the subjects see, hear or think of a number. For 11 other subjects (8 from the NF group), the representations arise only if contextual – mainly temporal – conditions allow it, and only 3 (2 from the NF group) subjects declared that the representations do not spontaneously arise. Moreover, 19 subjects declared they were unable to remove their representation once activated; only 5 subjects (4 from the NF group) can do this by concentrating upon something else (e.g., "upon the abstract operation being done") or even by "looking elsewhere".

Finally, the representations are *not very sensitive to the influences* proposed in the questionnaire: concurrent tasks seem to have little (reading, talking to someone, watching television) or no effect (driving a car, listening to music).

Distracting contexts keeping the subject's visual field free (like closing one's eyes, being in the dark) make the use of the representation easier (13 subjects). Moreover, conditions affecting the subject's concentration or general physical state (noise, fatigue, pain, etc.) seem to disturb the use, except for 2 subjects who reported using their representation precisely in these conditions, as they came "to help them".

Use of the number representations

Number representations and calculation

The number representations in calculation may, *a priori*, be used either to perform the arithmetic operation or more simply as a memory aid to visualize the data or some results of an arithmetical problem. We thus tried to distinguish these two aspects. All the subjects having an NF (except one) indicated that they visualized either the result or the elements of the operation or both.

Eleven subjects indicated using their number representation in the process of calculating, especially for addition (11 cases), sometimes for subtraction (6 cases) and even for multiplication and division (2 cases). However, the subject's explanations are quite ambiguous. For multiplication and division, no report indicated clearly which manipulations were done on the NF. Moreover, all subjects claimed to possess a normal knowledge of the "arithmetical facts" and in most cases to perform the calculations simultaneously in a classical way.

As regards simple addition and subtraction, four subjects gave detailed descriptions of some spatial operations realized on their NF. For example, one subject indicated that in order to realize "7 + 4", she first superimposes from the beginning of her NF a kind of grey transparent seven-step rectangle, then she transports a four-step rectangle (also constructed from the beginning of the NF) to the end of the seven-step rectangle and finally looks where the result is. She claimed to use a similar procedure for subtraction. She indicated, though, that she used such spatial procedures only when she was tired or when the retrieval of an arithmetical fact was not immediate. Another subject described the use of counting algorithms on her NF. When asked to add 4 to 7, she immediately enters her NF at seven, then climbs four units and looks at the result, realizing a mental "counting-on" algorithm. Another subject also described counting algorithms on the NF for addition and subtraction, while at the same time seeming to take advantage of some specific arithmetical facts. For example, if having to add 5 to 7, she explained that knowing that 7 plus 3 make 10, she immediately enters the NF at 10, "a vivid number easy to locate", then goes two steps further to "read the result". Subjects having analogical number representations also describe various calculation manipulations, which strikingly look like school learning procedures (displacement of dots to realize simple additions or subtractions). On the contrary, subjects having coloured number representations did not claim to specifically use that representation in calculation, and moreover, two subjects reported that in calculation contexts the numbers, if visualized, were not coloured!

Representation and other numerical contexts

Five subjects with coloured codes claimed using their representations in almost all the proposed contexts (especially for historical dates, addresses, appointment times, ages, games, etc.). The continuous scales are also used, especially for ages, appointments and historical dates. Nonetheless, some subjects of this group also said they rarely used them, or only in some special contexts. Regarding analogical representations, we do not have enough data to draw any conclusion.

Origin of the number representations

Eighteen subjects could not recall precisely the origin of their representation but proposed a date of emergence between 5 and 8 years, though one subject thought that his representations appeared at the age of 17. Some subjects had no idea (8), others thought their representation could be a direct (6) or indirect (5) result of a specific teaching method (2 subjects from the analogical group said it came from school teaching methods and the third from a personal calculation technique), while others thought no role had been played by environmental factors (7). None declared to have ever decided to construct such representations, nor had made systematic exercises to improve it. All subjects assumed that the representations evolved in parallel with the acquisition of numbers, but this seems to be a logical deduction rather than a precise recollection of specific autobiographical facts.

Presence of other representations

Seventeen subjects also presented visual forms or colours for hours, days, months, dates and/or the alphabet. In 8 out of 15 cases, the representation for hours is a clock face onto which the subjects have positioned some cues: either hour numbers (3, 6, 9 and 12 hours, for instance) or some personal information related to every-day routine activities. The days of the week are represented in 8 out of 14 cases by a horizontal graduated line, with each dash representing a day; other representations are a coloured code, circle or dial, or a strip of boxes. Here too, subjects may add some personal information. The months were represented either on graduated lines, circles or ellipses, sometimes with different colours for some periods of the year (see Figure 7).

Since only subjects having a representation for numbers were asked if they also had other visual representations, it is not possible to assess the frequency of

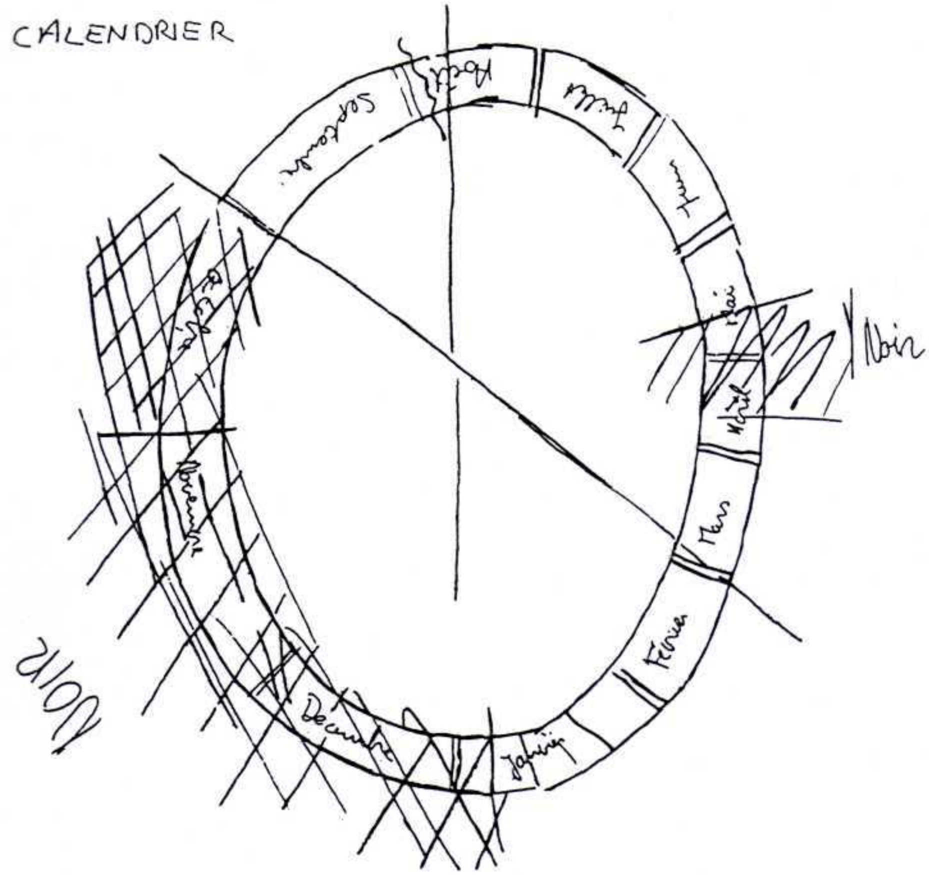

Figure 7. *Months' spatial representation for one subject.*

associations between those different representations. But given the relatively high percentage (61.5%) of subjects having both kinds of representations, it seems legitimate to postulate that they represent different aspects of a common phenomenon. If five subjects present hour, day or month representations of the same type as their number representation (see Figures 8 and 9), however, there exists no intra-subject homogeneity with respect to the structural characteristics of their visual forms. For example, one subject having elaborate NF for numbers, hours, months and dates had a coloured code for the letters of the alphabet; two other subjects had a non-elaborate NF together with a coloured code for numbers.

Presence of NF in other members of the family

Six subjects knew other members of their family possessing some number

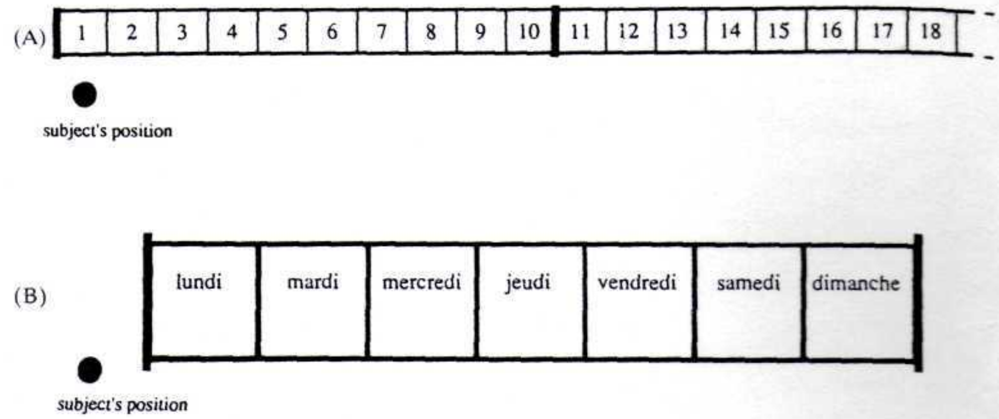

Figure 8. *(A) Number-form, and (B) days' spatial representation for one subject.*

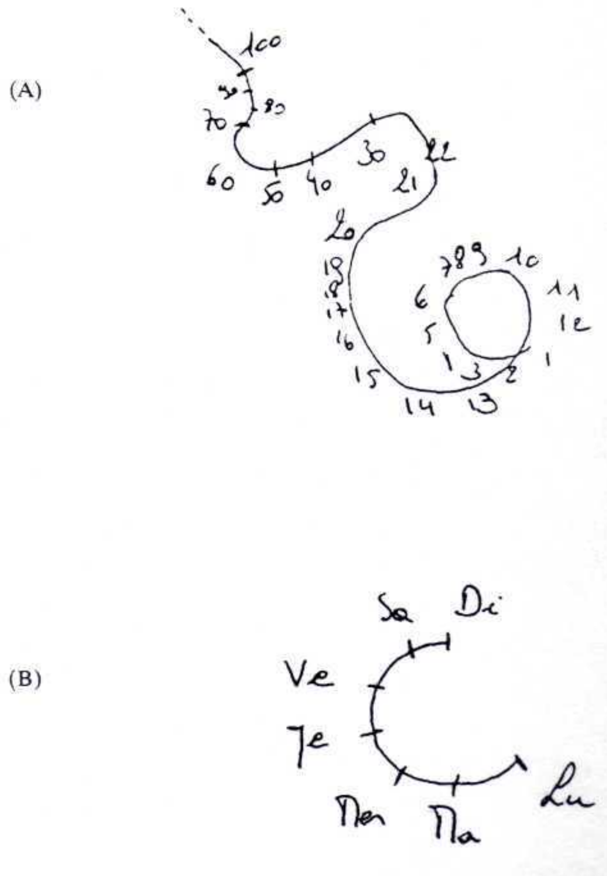

Figure 9. *(A) Number-form, and (B) days' spatial representation for another subject.*

Table 2. *Mean (m: I/23; V/30) and standard deviation (S.D.) of the imaginal and verbal scores*

		Norms[a]	Personal data
Imaginal score	m	18.4	19.23
	S.D.	(2.8)	(1.96)
Verbal score	m	22.0	17.54
	S.D.	(3.6)	(5.8)

[a]Paivio and Harschman (1983).

representation. Fathers, mothers and brothers or sisters were each cited twice. These "parent representations" were of the same type for only two cases.

Imagery

When compared with a control group of more than 700 students (Paivio & Harschman, 1983), the imaginal scores appeared to be only slightly higher ($t_{724} = -1.5$; n.s.), while the verbal scores were significantly lower ($t_{724} = 3.9$; $p < .01$). These subjects are thus not high imagers, but use particularly few verbal strategies (see Table 2).

Reliability of the description

All the subjects were questioned twice: first through the informal investigation or through the short questionnaire, then (more than two months later) with the extended questionnaire. When subjects were questioned for the first time, they were not told they might have to draw or describe their NF once again later. Furthermore, eight of the NF group subjects were asked to draw their NF one year after the first investigation. For the coloured code, the descriptions were strictly similar. Concerning the NFs, although three subjects introduced minor changes (small orientation shifts) in their representation, the general structure was always identically reproduced.

Summary

From a phenomenological point of view, our introspective data manifest an evident diversity: *synaesthesias*, *colour associations*, *analogical representations* and *NFs*.

The subjects that experienced synaesthesias did not report seeing numbers, but some numbers or digits evoked an emotional state or a sensation in another modality. Synaesthesias have been described in many others contexts (music: Rizzo & Eslinger, 1989; words: Baron-Cohen, Wyke, & Binnie, 1987) and thus are not solely related to numbers (Marks, 1978). Analogical representations were used for simple calculations (addition, subtraction, fractions) in a way that reflected some teaching procedures used in primary school. The colour association to numbers, or more generally to semantic linear orders, consist in viewing numbers, letters or other items of a semantically ordered class in a specific colour. This association is immediate and automatic, and for a given subject the same colour always appears with the same item.

NFs representing the succession of numbers in a spatial frame are the most interesting phenomenon, given that some of their characteristics are directly related to specific properties of the number system. Upon initial consideration, NFs show some common characteristics, with the first four matching Galton's constants: (1) they remain the same in time; (2) they seem to have emerged during childhood, with few identifiable relations to specific teaching methods; (3) automatic activation seems to be the rule; (4) the whole NF is not equally vivid, but when the subject concentrates on a number, its vividness increases; (5) numbers are represented in the digital code and arranged in their ordinal order, with some spatial characteristics depending on the base-ten organization of the arabic number system; (6) NFs are activated in arithmetical as well as in non-arithmetical contexts, but the frequency of their activation varies according to subjects and contexts; and (7) NFs are mainly used to visualize numbers.

PRELIMINARY EXPERIMENTAL INVESTIGATIONS

Number representations reported by some subjects were described in the previous sections. However, such introspective reports are clearly insufficient to attest structural or functional aspects of such representations, or even to prove their existence. Nonetheless, while one has to be wary of conclusions drawn only from introspective reports, we think (as Kosslyn (1980, p. 20) did when studying mental images) that they may be a valuable source of information which, when taken together with behavioural performance data, may assist the study of human representations. In this perspective, people who had described some mental representations of numbers were confronted with tasks implying these representations. We wanted to see if these subjects behaved the way they claimed, or the way predicted by their descriptions, or differently from people without any representation of this type.

The two main questions: "do these representations exist?" and "if so, are they the best explanation for the subjects' behaviour?" will of course not be totally answered with such preliminary investigations. Our present objective is only exploratory and certainly more data will be required in the future.

We shall limit the present investigation to two cases: one (GI) claiming to have a coloured code for digits, the other (JB) claiming to possess a scale NF. In the first case, our experimental investigation aimed to attest the genuineness of the coloured code, and in the second, to understand some functioning aspects of the use of the NF. Both cases are young people who were selected because they had agreed to participate. The subjects knew that the experiments concerned their number representations, but received no indications of the precise objective of each specific experimentation. A debriefing session was organized only when all the experiments had been administered.

Case 1

GI is a right-handed 23-year-old female finishing undergraduate school. Her number representation is of the coloured type where each digit from 1 to 9 is written in its own permanent colour (see Figure 6). GI reports that, confronted with a number, she immediately sees it written in its own colour. For two-digit numbers, there is no overlapping or blending of the colours, rather each digit is seen in its own colour (e.g., 62 is perceived as "sky blue plus green"). The colours are not verbally coded (the subject does not think "sky blue plus green", but sees the digits directly in their colour). GI says that the coloured code automatically forces itself in arithmetical contexts (visualization or memorization of intermediate or final results of an arithmetical operation) as well as in daily life contexts (memorization of numerical information such as phone numbers, car licence plate numbers, dates, etc.), and that it is not sensitive to external influences. GI also has coloured representations for the days of the week and the letters. She does not remember when all these mental representations first appeared.

GI gave us the colour she associated with each digit. While being mentally very precise, she said that some colours were not easy to label because they had specific shades. Experimental investigations addressing the consistency of her representations will be briefly described. Based on GI's report, it was predicted that: (1) if her code is personal and not easy to describe in words, the actual colours GI would select to illustrate her representation would be different from those selected by control subjects on the basis of her verbal reports; and (2) if her code is constant, GI should be able to reproduce it at any time in the same way.

Experiment 1: Production of the code

If GI's code is personal and not easy to describe in words, the actual colours she would select to illustrate her representation would be different from those selected by control subjects on the basis of her verbal reports.

Subjects

GI and and 10 control females with no mental number representation (mean age = 23 years; S.D. = 1.3) were asked to perform the task.

Material and procedure

Fifty pencils of different gradations of colours (mixable in order to obtain fine colour distinctions) were randomly presented to the subjects. The code indications were removed so that they could not be used as references.

GI were asked to pick out pencils in order to produce as accurately as possible the colours corresponding to her mental representation of the numbers from 1 to 9. The digits 0 and 5 were not included in the task because zero was described as colourless and five was associated with white in GI's representation.

The control subjects were given GI's description of the code (1 = black, 2 = green, 3 = red . . .) and were asked to select the pencils corresponding to the written colours.

The subjects were allowed to blend the colours to find the most correct shades of the target colour.

Results

GI performed the task easily: she mixed two pencils for the numbers four and seven and took only one pencil for each of the six other numbers. Control subjects, though given colours close to GI's, rarely chose exactly the same shades. We calculated the number of items for which each subject had chosen exactly the same pencil(s) as someone else. GI took exactly the same pencils as control subjects in only 17/80 (21.2%) cases. On the contrary, control subjects had much more agreement in their choices: the number of agreements ranged between 30/80 and 45/80, with a mean of 37.7 (S.D. = 6.27). When compared to 17 (GI's rate), this mean is significantly different (student t test: $t_9 = 10.44$, $p < .0001$).

Discussion

GI made specific choices that were different from the other subjects'. In other words, the agreement within the control group was higher than the one with GI. This could mean that GI's descriptions are not sufficiently precise to enable a person to select the colour corresponding exactly to her mental representations. This interpretation supposes that GI was actually performing the task on the basis of her representation, rather than on the basis of her precise labelling of the colours associated with each number.

Experiment 2: Reproduction

The second experiment was aimed at testing the long-term consistency of the colour production. If GI actually performed the production task on the basis of her internal coloured representation of numbers, then we should expect her to be able to reproduce the same choices later on, with maybe some variation in the choice of shade (i.e., shade errors). On the contrary, control subjects would not be expected to be able to do so, since they would do the task on the basis of their episodic memory traces of the production task only. They were thus expected to produce more errors. Furthermore, these errors would not be limited to the choice of the shade, but would also include errors such as producing one colour for another or producing a colour for one number that was actually associated with another number and so on.

Subjects

GI and 10 control females with no mental number representation (mean age = 23 years 6 months; S.D. = 1.9) were asked to perform the task. Five subjects were given GI's code (Group 1), and the remaining five were asked to think up a new code (Group 2).

Material and procedure

The same material as in Experiment 1 was used. Subjects of Group 1 were given GI's description (1 = black, 2 = green, . . .) and were asked to pick out the pencils corresponding to the verbal description. Subjects of Group 2 were not shown GI's description, but were asked to freely associate a specific colour with each digit, and then to pick out the corresponding pencils. As to GI, she had already produced her code in the first experiment. Seven days later, subjects of both groups as well as GI were asked to reproduce (without any written description) their previous choices from memory. (That is, Group 1 had to choose the pencils corresponding to GI's code, Group 2 the pencils corresponding to their personal "number–colour association", and GI the pencils illustrating her number representation.)

Responses were considered as correct if the colour chosen in the reproduction task was exactly the same colour associated with that specific digit in the production task. Errors were distributed into four categories: (1) shade error – choosing a different shade of the correct colour; (2) attribution error – choosing a colour that would be correct for another number; (3) shade attribution error – choosing a shade of colour that would have been correct for another digit; and (4) intrusion error – choosing a colour that had never been used.

Table 3. *Percentage of each kind of response*

Kinds of response	GI	Group 1	Group 2
Correct choice	75	5	55
Shade error	25	7.5	17.5
Attribution error	0	62.5	12.5
Shade attribution error	0	25	10
Intrusion error	0	0	5

Results

Table 3 shows the percentage of the different types of response for GI and the two control groups. Overall, GI was more correct than the control groups, but this difference was much less clear when compared to Group 2. As regards the type of errors produced, GI only made shade errors, while the two control groups also produced attribution, shade attribution or intrusion errors.

Discussion

As predicted, GI produced less errors than the control groups (especially Group 1) and these were only "shade errors". On the contrary, control groups produced much more and much more various types of errors. It should nonetheless be noted that GI's profile looked much more similar to Group 2's than to Group 1's.

Conclusions of Experiments 1 and 2

In the first experiment, GI produced a code corresponding to her verbal description. Control subjects confronted with GI's descriptions also produced colours corresponding to the description, but they did not choose exactly the same shades as GI did. Moreover, the proximity of the colour choices was greater between all the controls considered two by two than between GI and any of them. This result indicates that if the controls were guided in their colour selections by the verbal reports they had been confronted with, GI's colour selection was guided by another source of information that might be her internal representations.

At the reproduction task, subjects of the control groups produced more errors than GI. This was particularly true for those who, at the production stage, were asked to choose the colours corresponding to GI's description. Furthermore, subjects from the control groups produced not only shade errors, but also intrusion and attribution errors, which never happened for GI, who made only two subtle shade errors. This could be interpreted by the fact that at the reproduction task GI's selection continued to be guided by the same mental representations, whereas control subjects tried to recall the specific choices they had made one week before.

Nonetheless, the higher resemblance observed between GI's and Group 2's profile by contrast with GI's and Group 1's profile opens the door to an alternative explanation. The better scores observed in the reproduction task for Group 2 could reflect the existence of a non-random choice of the colours in the production task. Subjects may have chosen the colours according to a personal strategy (e.g., one subject chose a phonetic strategy for the first two numbers: "un → brun, deux → bleu", and chose the colour of the Belgian flag for the three following numbers), and this strategy could have helped them in the reproduction task (e.g., the subject mentioned above produced only one shade error in the reproduction task). A similar reasoning could apply to GI, even though she did not mention having developed such a strategy and was unaware of the way her representations appeared. In that case, the results of the first experiment could be explained by the lack of a detailed description of the colours (i.e., according to GI, 2 is "green" rather than leaf green, 3 is "red", rather than dark red, and so on). Such a vague description may explain the high congruency within the control subject's choices, by contrast with their low agreement with GI's. So, the question of a built-in number–colour association rather than a special coloured number representation remains open. However, in both alternatives, GI's results seem to prove that she had elaborated an association of the visual type since her choices cannot be explained by the verbal associations she produced between a digit and a colour.

Case 2

The second set of experiments was carried out with JB, a right-handed 17-year-old male, finishing high school. JB described his NF as an infinite vertical scale composed of many rectangular boxes (see Figure 4B) with each box containing one number written in digits. Positive numbers are towards the top and the negative towards the bottom. Every box has the same rectangular form but by an effect of perspective they become smaller when located far from the viewing point. The scale is also colour-contrasted: numbers are written in black on a white background (black numbers), except for numbers 11 to 20, and recursively, 111 to 120, 211 to 220 . . . , which are written in white on a black background (white numbers). Decades are more salient and cued by a horizontal line. To see a number in the scale, JB first visualizes the whole scale and then uses some sort of "zooming" procedure to locate a sub-part of the scale and then the specific number. In this procedure, JB explained the use of the decade-salient boundaries. For example, to locate 167 he uses 160 and 170. When looking at a single number up to 100, he generally sees together and in both directions (up and down) two or three adjacent numbers which are a bit less salient. However, higher in the scale, the surroundings are no longer visible and JB only sees salient numbers (such as 1300 and 1400 for the number 1372). When having to go from one number to another, he does not feel as if he is moving from one position to another one, but rather like making a "jump"! JB said that he could see two numbers together in the scale with the scale between them (e.g., 4 and 32), but the scale becomes fuzzy and discontinuous as the distance between the two numbers increases (e.g., 123 and 1000). JB reported going at the same speed when having to find two numbers in succession no matter whether the second number was higher or lower than the first one. JB also had mental representations for days and months.

We did not address JB's use of the NF for calculation in the experimental investigation. The following experiments aimed at testing if the time necessary to visualize a number in the NF corresponded to some of the introspective reports of JB. In this perspective we engaged JB in imagery tasks using a methodology close to Kosslyn's (1980). Numbers were presented visually to JB, who was asked to press a key as soon as he "saw" the number in his NF.

Experiment 1

The first experiment tests three comments reported by JB that could be summarized as follows:

(1) JB reported being able to locate, in quasi-single steps, numbers from 0 to 100, while using some kind of a zooming procedure for higher numbers. It was thus predicted that small numbers should be visualized more quickly than large numbers.

(2) JB indicated that when seeing a number in his NF, he simultaneously saw the adjacent numbers. On the contrary, he had to make "a jump" to go from one number to another one located further away. On this basis, it was predicted that, when asked to go from one target number to another, JB would take less time if the second number was adjacent or near to the first visualized number than if it was a number located further away.

(3) JB reported that "ascending" and "descending" jumps were equally easy to do. It was thus predicted that when asked to visualize two numbers in a row, the direction of the "jumps" should not have any influence on the speed of processing.

Material and procedure

JB was presented 64 pairs of stimuli (only black numbers were used) appearing in succession on a Macintosh SE TV screen. The following sequence of events had been programmed: the first number of a pair was presented, and JB had to push a keypad when he visualized that number in his NF (first reaction time: RT_1); a second number was immediately presented after his response and the subject also had to press a key when he visualized it in his NF (RT_2). In order to wash-out the subject's visual short-term memory between each trial pair, a visual interfering task was presented: a figure having the form of a V occurring in one of four different orientations (0°, 90°, 180°, 270°) was presented, JB had to answer up, down, left or right according to the direction toward which the V pointed. Out of the 64 pairs, there were 32 pairs of small numbers (from 2 to 30) and 32 of large numbers (from 102 to 2500). In each type of pair, 16 numbers were close to each other (distance $\leqslant 3$) and 16 were distant from each other (distance $\geqslant 15$). Finally in each sub-group of pairs, the second number was eight times above (ascending pair) and eight times below the target number (descending pair).

Results

Analysis of RT_1. Visualization times were significantly shorter for small numbers than for large numbers (mean 1092.12, S.D. 311.87 for small numbers; and mean 1440.09, S.D. 388.76 for large numbers; $t_{62} = -3.95$, $p = .0001$).

Analysis of RT_2. A three-factor ANOVA (Number's magnitude (small/large) × Distance (close/distant numbers) × Direction (ascending/descending from the first to the second number)) was calculated. There was a direction effect ($F_{1,56} = 4.919$; $p = .03$): Second numbers were more quickly visualized if located above the first one. We also had a clear distance effect ($F_{1,56} = 61.46$; $p = .0001$), no effect of the number's magnitude ($F_{1,56} = 1.39$; n.s.) but a significant interaction between these two factors ($F_{1,56} = 6.91$; $p = .01$). This interaction indicated that the number's magnitude had no influence on close numbers, but had an influence on distant numbers, the second numbers being visualized more quickly if they were small numbers than it they were large numbers (see Figure 10).

Analysis of RT_2-RT_1. A two-factor ANOVA was computed on the difference between the second and the first reaction times. Two factors were taken into account: the number's magnitude and the distance within the pair. Distance was the only significant factor ($F_{1,60} = 32.95$; $p < .0001$): for pairs of close numbers, the second reaction time was about 734 ms smaller than the first reaction time. By contrast, for distant pairs, the second reaction time was about 334 ms larger than the first reaction time.

Discussion

The time necessary to visualize a number is estimated by the first reaction time.

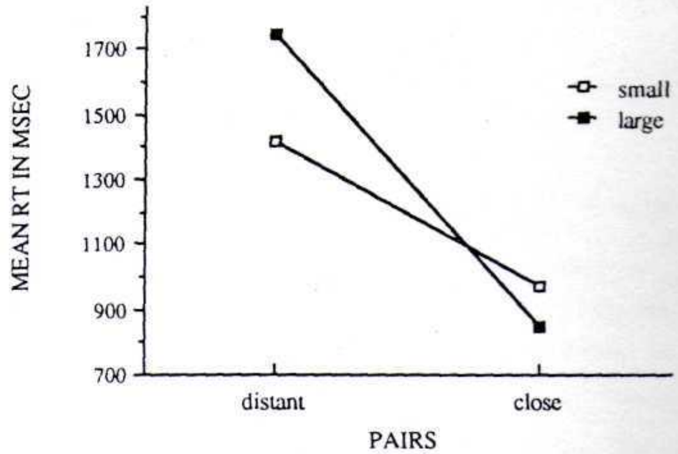

Figure 10. *Mean RT_2 according to the number's magnitude and distance within the pair.*

On average it takes JB about 350 ms more to visualize a large number than a small number, which supports our first prediction.

As regards the visualization of a second number, the time required is much shorter if this second number is in the neighbourhood of the first one (close pairs) than if it is distant from it. Moreover, those types of pairs seem to engage two different processings.

Close second numbers do not show a number magnitude effect, and the time necessary to visualize these numbers is shorter than the time necessary to visualize the corresponding first number. One may thus postulate that the visualization of the second close number is not accomplished through a second entry in the NF (since there is no number magnitude effect) but by a displacement of the attention from the first number to the second one. This disengagement of attention would be independent of the location of the numbers in the NF (since it occurred from the point in the NF where the subject was already positioned) and would be faster than the entry operation in the NF.

On the other hand, there was a number magnitude effect for distant second numbers (as for an entry operation). Furthermore, the time necessary to visualize those numbers was longer than the time required to visualize the first corresponding number. One may thus postulate that when confronted with a second number far from the first one, there would be a first look at the surroundings, followed, if the target number is not located in these surroundings, by a new entry operation which is (as seen for RT_1) influenced by the number's magnitude. The location of a second distant number would take more time than the location of the corresponding first number because the entry operation would be preceded by a checking of the surroundings.

Finally, and contrary to the subject's report, we found an effect of the direction, with a facilitation for ascending movement as compared to descending movement.

Experiment 2

The second experiment looked at the effect of the colour characteristic of the NF on the time necessary to visualize a number. Since JB had not been questioned on this point, no precise predictions were possible. This experiment also aimed at replicating some effects obtained in Experiment 1.

Material and procedure

The procedure was similar to the one used in Experiment 1. Sixty-four pairs of numbers were presented: 32 pairs of small numbers (from 2 to 30) and 32 pairs of large numbers (from 102 to 2500). In each group, there were 16 pairs composed of two black numbers and 16 pairs of two white numbers (number magnitudes were matched in the two groups of stimuli). The distance between two numbers in a pair was always small (± 3), but the second number was located below the first in half of the cases and above in the other half.

Results

Analysis of RT_1. A two-factor ANOVA (Colour (black/white) × Number's magnitude (small/large numbers)) indicated a main effect of both factors: small numbers were visualized more rapidly than large ones ($F(1, 60) = 6.08$; $p = .01$), and black numbers were visualized more quickly than white ($F(1, 60) = 10.346$; $p = .002$). There was no interaction.

Analysis of RT_2. An ANOVA with colour (black/white), magnitude (small/large numbers) and direction (ascending/descending from the first to the second number) indicated no effect of any factors or interactions.

Discussion

This second experiment confirms the number magnitude effect found in Experiment 1: small first numbers are visualized more quickly than large ones but this effect disappeared for the second number. The colour influenced the visualization of the first numbers: black ones were visualized more quickly than white. This effect also disappeared for the second number. This finding could indicate that while the subject was visualizing a sub-part of the scale the disengaging of attention to a close number was not influenced by the local colour of the NF.

Experiment 3

JB indicated that in order to see a number in his NF he was using the decade boundaries as cues, and then, through some kind of zooming procedure, found the correct number. This report was tested through a number visualization task in which numbers located near a decade frontier (e.g., 39) or at the middle of two decade frontiers (e.g., 36) were presented. It was predicted that numbers near the decade frontiers would be visualized more quickly than numbers in the middle of the decades.

Material and procedure

Two series of 20 isolated numbers above "100" were presented on a Macintosh SE TV screen. In each series, half of the stimuli were near the decade boundaries (± 1) and half at the middle. JB had to press a keypad as soon as he had visualized the number in his NF. RTs were measured.

Results and discussion

Numbers near the decade frontiers were visualized more quickly than those located at the middle of two decades (t_{38} one-tailed = 1.701; $p = .045$).

The role played by the decade boundaries to locate a number in the NF was thus supported by the results.

Conclusion of Experiments 1, 2 and 3

The results of the three experiments confirm many of JB's verbal reports about the structure and functioning of his NF, but disconfirm at least one.

The first confirmation is that JB takes less time to visualize numbers between 0 and 100 than upper numbers. Such an effect is reproducible since it is present in two experiments (1 and 2). Interestingly this effect is only present on the first number of the pairs and one may thus hypothesize that JB had to re-enter his NF. These effects fit well the hypothesis of two different access mechanisms: one direct to the distinct part of the NF (from 0 to 100) and one requiring a more complex and time-consuming procedure to locate a number in the less clear part of the NF, for example by first entering at a decade frontier then by zooming to a precise location.

A second main chronometric confirmation of JB's verbal reports is the fact that he visualizes more quickly the second numbers of close number pairs than those of distant number pairs, which may indicate that close numbers could be viewed without a second entry in the NF whereas such a second entry operation would be required for distant numbers.

A third confirmation of JB's verbal reports concerns the expected cueing role of the decade frontiers in the NF entry operation: JB actually visualizes more quickly numbers located near the decade boundaries than numbers distant from these.

A fourth result not directly expected from JB's reports, but compatible with them, is the fact that it requires less time to visualize a number in the small dark portions of his NF than in the white larger portions. One could interpret this result by suggesting that small black-coloured portions might play a cueing role with respect to entry into the NF and thus reduce the time required to localize a number. This effect is restricted to the first number of a pair, which suggests that it concerns only the entry operation.

The only results that disagree with some of JB's comments concern the above/below position of the second number in the NF. According to JB, it is equally easy for him to go from the bottom to the top as to do the reverse. However, the chronometric evaluation contradicts such an impression: going up takes less time.

One may, however, argue that all these data suffer from Pylyshyn's classic argument, namely that JB simply did what he had said he does, and that his behaviour, rather than reflecting operations on his NF, simply simulates what it would be like to use his NF for answering our questions. Yet this argument seems very improbable, since JB was totally ignorant of the objective of each experiment and of the structure of the material. Furthermore, JB is a young man uninformed about concepts and research in cognitive psychology and in mental imagery, and thus it seems very improbable that such a naïve subject could control such fine differences in reaction time. One may of course postulate that similar data could be gathered with control subjects having no NF. We tried to conduct such control experiments with six subjects who all interrupted the experiment after less than five minutes, claiming that this had no sense for them!

Finally, one may argue that the results obtained by JB may be understandable without postulating the existence of an NF. For example, given that the numbers from 0 to 100 contain fewer digits than numbers beyond 100, and given that simple images take less time to generate than more complex ones, small numbers could be more quickly visualized than larger ones. However, we do not know any plausible interpretation of some other specific effects such as the more rapid visualization of numbers in the black part of the NF or of numbers close to the decade boundaries. The most economic interpretation is thus that JB has effectively an NF, some of whose properties he had been able to introspect about.

GENERAL DISCUSSION

Genuineness, or what is true?

The primary question remaining open is the genuineness of the number representations. It could be argued that the subjects had imagined possessing such representations, or that they had told us some interesting, but false, stories about an aspect of their mental life. Yet the genuineness of these representations could be supported on the basis of the structural similarities existing between several reports from different inquiries done in various countries (between-subjects reliability), as well as on the intra-subject stability of introspective reports.

Concerning the between-subject consistencies of NFs, the similarities existing between the descriptions of Galton (1880a, 1880b, 1883), Bertillon (1880, 1881, 1882), Spalding and Zangwill (1950) and our own data are so numerous that it seems very unlikely that subjects could have independently invented specific structural characteristics such as the saliency of the decade boundaries, the opposition between clear (near the origin) and fuzzy (far from the origin) sub-parts, the up and/or right location of positive numbers in comparison with bottom and/or left location of negative numbers, or in some subjects the clock-like organization of the first part of the NF.

Regarding colour association, the same between-subject stability argument applies. Aside from our experimental demonstration of intra-subject consistency (Case 1), we investigated whether there was some between-subject stability in colour association. As suggested by Shanon (1982), between-subject stability does not imply that associations are the same for all people since some subjects have no colour association for each digit and others associate one colour with more than one item. According to Shanon, the stability of colour association is a characteristic of entire patterns, not of individual items. From an inquiry involving 18 subjects on numbers from 0 to 10, Shanon calculated the mean numerical value for each colour and observed that black and white colours presented a bimodal distribution; that is, they were associated with the extreme (high or low) values on the numerical scale. He also observed that the other colours were not randomly distributed across numbers but appeared in an organized succession that mimicked their probability of appearance in different languages (Berlin & Kay, 1969)[2]. The same analysis was applied to our own data (11 subjects; see Table 4). Like Shanon, we found a clear bimodal distribution for the white and black colours, and, even if our hierarchy of colour association is not strictly identical, the correlation between colour order in the two studies is highly significant (Spearman rank order correlation, $N = 12$, $r_s = 0.88$; $p < .001$). We also compared, as did Shanon, the colour association with the linguistic hierarchy of colour proposed by Berlin and Kay (1969) according to the colour probability of appearance in various languages. A positive and significant correlation between the mean numerical values of our colour association and the linguistic typology

[2]Berlin and Kay (1969) have established that if a colour term is found in a given language, all colour terms that precede it in the order defined in the typology will be found in that language. The typology stipulates that the first colour distinction is between white and black, then red is introduced, followed by yellow and green, then blue and brown, with a further distinction being made between orange, purple, grey and pink.

Table 4. *Mean numerical values corresponding to colour associates with numbers, from Shanon (1982) and our personal data*

	Shanon (1982)	Personal data
Black	0.62	1
White	0.63	1
Yellow	3.18	5.7
Red	4.73	4.9
Blue	4.89	6.6
Green	5.05	4.6
Orange	5.33	5.5
Brown	6.86	7.3
Violet	7.21	9
Grey	8.50	7.7
Black	8.92	8
White	9.50	9

was also found (Spearman correlation of ranks; $N = 10$, $r_s = 0.88$; $p < .01$; Shanon obtained a coefficient of 0.92). It seems then that our between-subject stability shares some general characteristics with those evidenced by Shanon and it is very improbable that such regularities are solely attributable to individual inventions!

Intra-subject stability of the NF constitutes the second main argument. The results we obtained on a group of subjects by using a re-test procedure showed, except for structural details, a high consistency between the two descriptions for NF and a perfect consistency for colour association.

However, in order to reinforce the genuineness claim, more direct experimental evidence has to be collected in the future. At present, we have only experimental data on the stability of the colour code for one subject (Case 1) and chronometric data that were predictible from (or at least not incompatible with) some functioning aspects of the introspective reports in another (Case 2). It is surely not enough, but these results are encouraging.

In the following discussion, given the spatial characteristics of the NFs closely related to some fundamental aspects of the numbers system, we shall concentrate on these and pose four main questions: (1) Are the so-called "NFs" specifically related to numbers? (2) Are they different from other visual representations of numbers? (3) Are they actually used in calculation? (4) What possible explanation of the origin of the NFs could be suggested? Finally, we will briefly speculate about the psychological relevance of the NFs.

Specificity

The fact that some subjects possess stable spatial representations for elements other than numbers (hours, days, dates, months, letters of the alphabet (Bertillon, 1980; Spalding & Zangwill, 1950)) could be a main counter-argument for the numerical specificity of the NFs. However, it is remarkable that NFs are associated with elements that share some striking similarities with numbers: they constitute well-delineated sub-parts of the lexicon, are sequentially organized and have been learned by rote in a conventional order during childhood. It is noteworthy that these properties also characterize the items related to colour associates (Shanon, 1982). Furthermore, some of our subjects have developed both systems of association, and similar co-occurrences have also been described elsewhere (Baron-Cohen, Wyke, & Binnie, 1987). The possibility must thus be considered that NFs and colour association have a common origin and present some functional similarities.

NFs compared to other representations of numbers

Many people can, if required, visualize numbers as one would write them to perform an arithmetical operation. There also exists strong evidence that Japanese subjects calculating with an abacus visualize on a "mental abacus" the moves corresponding to the operations they really perform on an actual abacus. Moreover, some calculating prodigies have been reported to use visual strategies in calculation. One may thus ask whether there are some structural or functional similarities between those visualizations of numbers and the NFs here described. We think this is not the case, but until now our arguments were based only on indirect evidences.

The general capacity everyone possesses to mentally visualize a number or an arithmetical written operation as they would appear if they were actually written has been brought to light in many studies (Hadamard, 1945; Skemp, 1971; Syer, 1953). This capacity may play a short-term memory role in mental arithmetic (Hitch, 1978) and mathematical problem solving (Hayes, 1973). However, this kind of imagery is nothing more than the capacity to visualize the elements given in the problem and the classical steps of written resolution, and shares no evident relationship with an automatic activation of a stable spatial representation of numbers.

Similarly, skilled abacus users were described as visualizing a mental image of the abacus, and performing rapid mental calculations by manipulating the beads on their "mental abacus" (Hatano, Miyake, & Binks, 1977; Hatano & Osawa, 1983; Hatta, Hirose, Ikeda & Fukuhara, 1989; Stigler, 1984), but this is not a standard spatial representation of the numbers. The subjects' representations are, however, related to the specific problem they are confronted with, and remain completely dependent on extensive practice calculating with a real abacus (or at least, practice of the specific abacus calculation technique (Enerson & Stigler, 1986)), which is not the case with NF.

The case of prodigious calculators needs to be more closely examined since some of them were described as presenting synaesthesia, colour association and even NFs (Bertillon, 1880). Visual calculators have been divided into two classes (Smith, 1983, 1988): those seeing numbers in their own handwriting, and those having elaborated a visual image of numbers they are confronted with. In the latter category, some visual calculators have been described as being able to visually retain many numbers presented on a sheet of paper and, if the numbers were coloured, to memorize simultaneously the associated colours. Others indicated that in the process of calculation they also frequently generated images of numbers in their own handwriting. Yet, whatever the intrinsic interest of these observations to understand the processes that sustain specific talented abilities, these visualizations seem to be very different from the NFs we have described. As a matter of fact, calculating prodigies did not indicate that they saw numbers in the same place in an invariant NF; they only reported seeing numbers in an isolated way or seeing the series of numbers they have created during the process of calculating. In some cases the numbers visualized by genius calculators are also spatially organized but this organization is related to the specific algorithm of resolution they use. With respect then to the genius calculator having an NF and who was able to mentally perform extremely complex calculations such as multiplying two 15-digit numbers (Galton, 1880a), he reported that when performing complex multiplications he did not use his NF. Furthermore, our subjects neither claimed to be especially competent in calculating, nor showed an extended and rich semantic knowledge of number configurations or of computation strategies. Finally, whereas expertise in calculating is always linked to an extensive (in some cases, compulsive) practice of numbers (Smith, 1983), none of our subjects reported to have spent a great amount of time playing with numbers or visualizing their NF. However, this practice argument is not entirely convincing given that, if the NF is frequently activated by any heard, viewed or thought number, then it is extensively used!

Thus, there are not many structural or functional similarities between the representations used by the prodigious calculators, the expert abacus users and normal subjects and the NFs, except, of course, that they were of a visual nature.

Calculation and NFs

Another important point is to clarify the role of the NFs, if any, in calculation. The data we have collected are, however, not completely convincing. Of course, some subjects indicated using their NF to carry out some calculations, but the fact that they were also able to perform the calculation by accessing arithmetical facts directly in a non-visual way is not easy to interpret. Our protocols contain two different types of reports: in the first type, the subjects only indicated that some numbers are activated while calculating; in the second one, subjects described calculation procedures they directly realize on the NF.

Activation of numbers during calculation

When the subjects indicated only the activation of some numbers on the NF while calculating, such activations may only represent the output in working memory of a calculation performed on a verbal or abstract code. However, other subjects describe the activation of an area in their NF *before* they recovered the result of their calculation, and such a temporal pattern could indicate that the NF is used for a global estimation of the result. In this connection, two subjects indicated that, when confronted with an operation, they knew approximately in which area on their NF the result should be located (this part of the NF became a little more vivid) and that, when the operation was realized, the result become more salient. However, such a description may either correspond to a general mode of access used by the subject to locate a number on his NF, or it could also be that the sequence "first an area more vivid, second the isolation of the precise result" corresponds to a two-step calculation process but that the estimation operation has also been realized in another data structure before activating an area in the NF. Finally, it is very hazardous to be confident about the timing of mental processing derived from introspective reports only.

Manipulation of the NF during calculation

Four subjects described in some detail the mental operations they did on their NF when calculating. One subject described how she realized simple addition and subtraction by displacement of segments on the NF, in a way very similar to that hypothesized by Restle (1970) on the "number-line"; another described counting-on algorithms frequently used by children confronted with concrete collections of objects (Fuson, 1982). Such similarities are interesting but, of course, do not constitute scientific proof.

Finally, some arguments in favour of a role of the NF in calculation have been suggested in one neuropsychological study. Spalding and Zangwill (1950) presented a 24-year-old patient who had suffered a left-sided occipito-parietal brain injury resulting in severe dyscalculia, disorders in visual memory, some impairment of topographical sense, and, interestingly, the loss of a previously well-developed NF. After the lesion, the patient indicated that his NF had lost its former distinctness and could not be held in mind for purposes of calculation. The patient was slow and often wrong when performing both oral and written two-digit additions, subtractions and multiplications. The actual significance of the

NF disturbance in relation to the dyscalculia remains, however, far from clear, especially because the patient also presented a residual aphasia, and the latter might explain the difficulties in retrieving arithmetical facts.

Thus at present there exists no clear evidence about the role of the NFs in calculation. We think it represents a point of critical importance to be clarified in the future. The main way to solve it should be to submit subjects who claimed to use NF in calculation to classic chronometric studies on production or verification of simple arithmetical problems, in order to see if their reaction times present patterns similar to that of control subjects or some idiosyncratic patterns interpretable as the result of manipulations on their NF.

In doing so, one has also to examine the possibility that the NF may be used not for precise calculation but rather for global estimation of quantity. Dehaene (1989) has recently argued that normal subjects may use some analogical spatial processing to compare two-digit numbers. Moreover, he indicated that in the case of cerebral lesions such analogical processing may be preserved whereas exact calculation may be impaired (Dehaene & Cohen, 1991). Even if the model presented does not consider configurations other than a horizontal and asymmetrical straight line, it is striking that our subjects who reported estimating the result before seeing the correct answer also described a kind of focusing mechanism very similar to the one depicted in Dehaene's model. Of course, here too a simple epiphenomenal resemblance between a theoretical model and some subjects' introspective reports is insufficient to validate either the introspection or the model. In the future it would indeed be interesting to conduct some of the experiments designed by Dehaene with subjects having an NF to observe whether or not they use a "normal" number line.

Origin of the NFs

A last question is, of course, why some subjects have an NF while most people do not. It could be, however, that this is not a pertinent question, and that the difference is not of an all-or-nothing type. There are indeed many differences among our subjects concerning the vividness of their NF, its structural complexity and its frequency of use. Aside from elaborate NFs as described above, some subjects indicated possessing more elementary NFs – either a simple horizontal line centred on zero or a short vertical line up to twenty. Other subjects reported not having an NF but experiencing "a feeling" of number position, such as "fourteen is closer to fifteen than to ten" or "forty-two is between forty on the left side and fifty on the right". Similarly, Hunter (1957) indicated that among 250 people who claimed to have no NF, no fewer than 210 reported "a feeling that numbers somehow recede from them: some reported that numbers had a vague upward movement and others that they seemed to recede in a straight line or at an angle". It could thus be that the NF is only a more accomplished development of a general disposition to encode in a visual way some linear ordering. In this direction, it should be underlined here that Dehaene, Dupoux, and Mehler (1990) postulate the existence of analogical processing on a mental number-line in tasks requiring number magnitude comparison, and that they have observed a response side effect: the time to respond "larger" is shorter for a right location of the response, whereas the time to respond "smaller" is shorter for a left location of the response. Such a response side effect can be tentatively interpreted as a facilitation due to a spatial isomorphism between the location of the response and the location of numbers on the number-line, small numbers being located on the left side, large numbers on the right. In line with this, we also found such a defined orientation: 10 of the 15 collected NFs were left-to-right orientated, and only one right-to-left orientated. Moreover, a bottom-up orientation also appeared: in 9 (out of 15) NFs the numbers progressed from the bottom up (and only one was top-down orientated).

Whether or not there exists a continuum between less elaborate spatial representations and the NFs, the question of the conditions that favour the development of an elaborate NF remains open. Any satisfactory answer to the problem of the origin must of course be based on developmental observations, and given the absence of any empirical investigations on children, our commentaries will remain conjectural. The possibility of a genetic predisposition cannot be ruled out as yet in the present state of our knowledge. Galton underlined that NFs were more frequent in women than in men, but he had not conducted a precise statistical analysis. In the same way, our investigation of a student population presents many selection biases which render it impossible to establish the presence or absence of sex differences. Finally, the fact that six subjects reported that some members of their family also presented NF is not a sufficient argument, given that similarities in educational environment could adequately account for similarities in number visualizations.

Of interest with respect to the origin of the NF is, on the one hand, the results from Paivio's questionnaire, and, on the other hand, the fact that NFs were related to semantic linear orderings. Paivio's questionnaire indicated that our subjects were not specifically gifted in visual strategies, but that they were less efficient in verbal strategies. This then suggests that an elaborate NF, rather than being the result of an above-average competence in visual imagery, could be the result of some compensation strategies in subjects having difficulties with verbal learning by rote of linear orders. One could imagine that children who experience some difficulties learning verbally the sequence of numbers, letters of the alphabet, days of the week or months (all sequences which can only be learned through next-one relations) take advantage of a simultaneous visual coding of the order of the items, using spatial ordering to reinforce the temporal coding of the elements in the sequence. The compensatory model, proposing that greater abilities in one area could result in weaker abilities in another or that a weakness is compensated for elsewhere by a particular strength or talent, has been proposed in other areas by developmental psychologists (for second-language acquisition: Schneiderman & Desmarais, 1988; talented idiot savants: Hill, 1978; Boggyo & Ellis, 1988; Charness, Clifton, & MacDonald, 1988).

The significance of the NFs

In the preceding sections we presented introspective data as well as two experimental single-case investigations aimed at documenting the existence and some functional aspects of specific visual representations of numbers. One has, of course, to question the psychological significance of such phenomena. As an anonymous reviewer noted on a previous version of this paper, one may wonder if such NFs are no more interesting than the three-headed goat, the fattest woman on earth or the idiot savant! Although fascinating, NFs could thus be of no importance for a cognitive theory of human numbers representation. However, a first argument for considering NFs seriously is their frequency of appearance. Galton considered that 3% of males and 6% of females presented NFs; Hunter (1957), looking more generally at number representations, indicated that visual representations or sensorial experiences evoked by numbers were present in 210 out of 250 subjects; in our study on a group of students in psychology, we find a percentage of 14% of subjects reporting the existence of visual representations for numbers.[3] The phenomenon, although rare, is thus not exceptional, and is, for example, more frequent than left-handedness! Yet, even if frequent, NFs may well constitute a peripheral aspect of our cognitive functioning with no pertinence for the understanding of the basic aspects of number processing and calculation. At present, we have no solid data to reject or accept this argument but some hypotheses may be advanced to orientate future research. One has, for example, underlined that NFs are associated with specific semantic linear orders, and it must be remembered that the production of the conventional sequence of numbers is one of the most crucial competences underlying basic counting abilities, which constitute a central skill for the development of subsequent mathematical cognition in children (Fuson, 1982; Gelman & Gallistel, 1978). The NF could thus constitute a spatial medium in which number sequence is represented, maybe because of some difficulty encountered for the coding and learning of such a sequence by language alone. If so, NF may play a critical role in a basic aspect of arithmetical cognitive development, at least for some (to be identified)

[3]Moreover, another inquiry (not yet analysed and not included in these lines) on 217 students (91 females, mean age of 20.1) coming from a variety of departments (arts, philosophy, psychology, engineering, etc.) showed that 42 of them reported having an NF. A higher rate was observed among females (26.37%) as compared to males (14.3%).

subjects. It must be acknowledged that, at present, such a suggestion is purely speculative.

More broadly, NF may be considered as a new ingredient in the unresolved debate on the nature of number representations. Aside from the prominent view of unitary abstract number representations (see McCloskey, this issue, and McCloskey, Sokol & Goodman, 1986), alternative views of number representations have been put forward suggesting an "encoding-complex model" that proposes an integrated network consisting of many format-specific number codes and processes that collectively mediate number comprehension, calculation and production (Campbell & Clark, 1988). Even if this model lacks precision, some data suggest the existence of multiple levels of representations for numbers and underline the possible relations existing between specific number codes and number representations or calculations. For example, it has been proposed that stimulus format (roman or arabic) influences calculation processing (Gonzalez & Kolers, 1982), that multiplication facts are phonologically coded at least in Japanese (Kashiwagi, Kashiwagi, & Hasegawa, 1987) which could result in breakdown dissociations in cases of aphasia, and that the syntactical structure of some language systems (English by comparison with Japanese) may influence children in the understanding of place value of the arabic numeration system (Miura & Okamoto, 1989). Furthermore it has been proposed by Restle (1970) and more recently by Dehaene (1989) that numbers may be encoded analogically on a number-line which may be used for magnitude comparison of two-digit numbers. From the existence of some structural similarities (left to right progression of the number magnitude) between our NFs and the number-line, one may speculate about a possible continuum between those visual representations!

Our last conclusion is that NFs seem to exist and that they would specifically code linear orders including the sequence of numbers. Such a phenomenon should be examined in the future to establish its function, if any, in number and calculation processing during development and in adulthood.

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